Extraction of Modal Parameters Using Experimental & Operational Modal Analysis for Overhanging Machine Component

Beena Limkar, Dr. Gautam Chandekar

Abstract— Overhanging machine components are mostly susceptible to vibrations and hence are critical from design view point. These overhanging components can be generalized as a cantilever beam. Cantilever beam problem is taken as a case study in this paper as the analytical results are easily available. Most often the classical Experimental Modal Analysis (EMA) cannot be used in such cases as the exact operating and boundary conditions are critical for better accuracy in modal parameters extraction. In this paper, Operational Modal Analysis (OMA) is considered as a substitute for EMA. OMA uses data measured at the response points and the requirement of the excitation data is eliminated and hence the data can be collected under operating conditions. A comparative analysis of analytical results and classical EMA is presented. The datasets collected using hammer tips with different hardness are considered for extracting first five modes of cantilever beam using EMA. Based on this study, a particular hammer tip dataset is selected for each mode, which is further considered for OMA. Five different OMA techniques are established using this case study. This is followed by a comparative study of classical EMA with the five different OMA techniques.

Index Terms— Overhanging Machine Component, Cantilever Beam, Comparative Study, Experimental Modal Analysis, Modal Assurance Criteria, Operational Modal Analysis.

1 Introduction

MODAL ANALYSIS is fundamental for vibration analysis and forms backbone for advancements in vibration analysis. The overhanging machine components are susceptible to vibrations and thus become critical from design point of view. The simulation of the operating and boundary conditions, are critical for the accuracy of modal parameter extraction. The overhanging machine components can be generalized as a cantilever beam. Cantilever beam is used in this study due to the easiness in understanding many fundamental facts in vibration analysis. This paper studies the analytical and experimental techniques in modal analysis using a cantilever beam. Apart from classical Experimental Modal Analysis (EMA), a few algorithms in Operational Modal Analysis (OMA), Rainieri [1], are studied.

Many significant developments in OMA have started since 1990s. However, OMA was used during ancient times to understand the dynamic behavior of a system. It is well known that the Greek philosopher Pythagoras studied the origin of musical sound using strings. This technique was nothing but applied OMA, though it was not developed systematically then. Similarly, other scientists like Galileo, Daniel Bernoulli and Newton, who have also contributed their theories in the vibrations domain, have used OMA in their experiments in one form or other, Brincker [2].

The systematic development in OMA started around 1994 when the damage done due to Northridge earthquake was studied. Most of the development in OMA started with time-domain techniques such as NExT method. During the late twentieth century a comparative study of the peak-picking method, the polyreference LSCE and the stochastic subspace identification methods was presented in, Peeters [3]. The application of these stochastic subspace identification methods along with the NExT technique was presented in, Hermans [4]. OMA finds direct application in the structural health monitoring (SHM) and damage detection of the structures.

Vibration modes of a cantilever beam are used for identifying crack location, Rizos [5], Kumar [6]. Data obtained using laser vibrometer for a cantilever beam is analyzed using EMA, Sharma [7]. The dynamics of cantilever beam are studied by, Kane [8], for a dual purpose. Firstly for studying behavior of the rotor blades, spacecraft antenna, etc. and secondly to find flaws in existing multibody computer algorithms. Another study of rotating cantilever beam was carried to find the tuned angular speed at which the resonance occurs, Yoo [9].

This paper aims at a comparative study of the analytical and experimental methods of modal analysis. The results obtained using classical EMA are compared with the analytical results. The results obtained using OMA are then compared with the results obtained using EMA. In practice, analytical solutions are rarely available and hence EMA becomes important. In classical EMA the structure is excited using hammer or shaker, but this is a limitation especially in case of big structures. The boundary conditions used in the laboratory testing are different from those under working situations. Hence OMA give a solution to these problems as OMA is a technique which uses only response data for extracting the natural frequencies and the mode shapes, and thus replaces EMA under such situations.

2 Theoretical Background

2.1 Analytical Method

The analytical solution for the natural frequencies of a cantilever beam are computed as given in Inman [10],

ISSN 2229-5518

$$\omega = \beta^2 \sqrt{\frac{EI}{\rho A}}$$

The mode shape for the cantilever beam is expressed as,

$$U_n(x) = \sigma_n(\sin\beta_n x + \sinh\beta_n x) + (\cosh\beta_n x + \cos\beta_n x)$$

The values of σ_n and $\beta_n l$, where l is the length of the beam, are tabulated in Table 1.

Table 1: Values for Fixed-Free Beam

n	σ_n	$\beta_n l$
1	0.7341	1.87510407
2	1.0185	4.69409113
3	0.9992	7.85475744
4	1.0000	10.99554073
5	0.9999	14.13716839

2.2 Experimental Modal Analysis (EMA)

The frequency response function (FRF) is expressed as, Heylen [11],

$$H(\omega) = \frac{A_1}{i\omega - \lambda_1} + \frac{A_1^*}{i\omega - \lambda_1^*}$$

At resonance the complex conjugate part becomes negligible, therefore the FRF is expressed as,

$$H(\omega) = \frac{A_1}{i\omega - \lambda_1}$$

The imaginary component contains the modal information and the real component contains the damping information. At resonance the value of FRF is maximum and thus the maximum value gives the amplitude of the mode shape at a given location and corresponding frequency is the natural frequency.

2.3 Operational Modal Analysis (OMA)

2.3.1 Basic Frequency Domain Method

The Basic Frequency Domain (BFD) Method is based on the fact that at resonance only one mode is dominant. At resonance the structural response can be considered as the modal response, Felber [12].

$$x(t) = \phi_r p_r(t)$$

Where $p_r(t)$ is the modal coordinate and ϕ_r the mode shape for the r-th mode. BFD Method use the correlation functions. The power spectral density (PSD) matrix is given by,

$$G_{YY}(\omega) = G_{P_rP_r}(\omega)\phi_r\phi_r^{\ H}$$

The PSD matrix has rank one at resonance and any of the PSD matrix column give the mode shape. The BFD method gives greater accuracy when the damping is low and the modes are well separated. This is a useful tool to get a quick insight about effectiveness of measurements.

2.3.2 Frequency Domain Decomposition Method

The Frequency Domain Decomposition (FDD) Method, Brincker [13], is a useful technique to identify closely spaced modes. We have the PSD matrix given as,

$$G_{YY}(\omega) = G_{P_rP_r}(\omega)\phi_r\phi_r^{\ H}$$

The singular value decomposition (SVD) of the PSD matrix is given by

$$G_{YY}(\omega) = [U][\Sigma][V]^H$$

The singular value matrix $[\Sigma]$ gives the active natural frequencies in the response and the corresponding column of $[U]$, gives the mode shape.

2.3.3 Least Square Complex Exponential Method

Least Square Complex Exponential (LSCE) Method, Mohanty [14], is basically a curve-fitting algorithm for extracting modal parameters from the correlation functions. The measured response data is a discrete time data. The correlation for this data can be expressed as summation of decaying sinusoids.

$$R_{ij}(k\Delta t) = \Sigma_{r=1}^{n}(C_{ij,r}e^{\lambda_r k\Delta t}) = \Sigma_{r=1}^{n}(C_{ij,r}z_r^k)$$

Where

n = 2Nm = number of modes

$C_{ij,r}$ is constant for r-th pole

λ_r is the r-th pole

z = $e^{\lambda_r \Delta t}$

These correlation functions are used to formulate a Hankel matrix.

$$[R_{ij}] = \begin{bmatrix} R_{ij}(k\Delta t)_{k=0} & R_{ij}(k\Delta t)_{k=1} & \cdots & R_{ij}(k\Delta t)_{k=n-1} \\ R_{ij}(k\Delta t)_{k=1} & \cdots & \cdots & R_{ij}(k\Delta t)_{k=n} \\ \vdots & \vdots & \vdots & \vdots \\ R_{ij}(k\Delta t)_{k=n-1} & \cdots & \cdots & R_{ij}(k\Delta t)_{k=2n-2} \end{bmatrix}$$

Using a specific number of samples a set of equations is obtained.

$$[R_{ij}]\{\beta\} = -\{\overline{R}_{ij}\}$$

$$\therefore \{\beta\} = -[R_{ij}]^{-1}\{\overline{R}_{ij}\}$$

The β values are the coefficients of the Prony's equation

$$\beta_0 z_r^0 + \beta_1 z_r^1 + \cdots + \beta_{n-1} z_r^{n-1} + z_r^n = 0$$

Once these coefficients are known the roots of the Prony's equation, $z_r^k = e^{\lambda_r k\Delta t}$ are found. From these roots the natural frequency, the damped modal frequency and the damping ratio is obtained after conversion in Laplace domain.

The mode shapes are obtained using these values in equation for $R_{ij}(k\Delta t)$. LSCE method is thus a two stage method as the mode shapes are estimated only in the second stage.

2.3.4 Eigenvalue Realization Algorithm

In the Eigenvalue Realization Algorithm (ERA) technique, Juang [15], uses decaying samples in time. Each decaying time samples are represented as,

$$y(k) = PD^k u_0$$

The set of samples are represented as,

$$Y(k) = [y_1(k), y_2(k), \cdots]$$

$$U_0 = [u_{01}, u_{02}, \cdots]$$

$$\therefore Y(k) = PD^k U_0$$

These decaying time domain samples are used to form two Hankel matrices. These two matrices are just one time-step offset from each other.

$$H(0) = \begin{bmatrix} Y(0) & Y(1) & \cdots \\ Y(1) & Y(2) & \cdots \\ \vdots & \vdots & \vdots \\ Y(s-1) & Y(s) & \cdots \end{bmatrix} H(1) = \begin{bmatrix} Y(1) & Y(2) & \cdots \\ Y(2) & Y(3) & \cdots \\ \vdots & \vdots & \vdots \\ Y(s) & Y(s+1) & \cdots \end{bmatrix}$$

The observability (Γ) and controllability (Λ) matrices are obtained from these Hankel matrices.

$$H(0) = \Gamma\Lambda$$

$$H(1) = \Gamma D \Lambda$$

In Eigenvalue Realization Algorithm (ERA) technique, Juang [15], SVD of first Hankel matrix is taken so Γ and Λ can be expressed as,

$$H(0) = USV^T$$

$$\Gamma = U\sqrt{S}$$

$$\Lambda = \sqrt{S}V^T$$

Discrete time system matrix is estimated as,

$$D = \Gamma^{-1} H(1) \Lambda^{-1}$$

To simplify the inverse calculation, reduced matrices U_n, S_n and V_n are used where $n = 2 \times \text{number of modes}$.

$$D = S^{-1/2} U_n H(1) V_n S^{-1/2}$$

Using this discrete time system matrix the eigenvalues and eigenvectors are obtained.

$$D = [\phi'][\mu_n][\phi']^{-1}$$

The eigenvalues λ_n are then found using relation

$$\mu_n = \exp(\lambda_n t) \Rightarrow \lambda_n t = \ln(\mu_n)/\Delta t$$

The mode shapes need to be transformed as follows,

$$[\phi] = P[\phi']$$

Where

$$P = U_r S_r^{1/2}$$

2.3.5 Transmissibility-based Operational Modal Analysis

Transmissibility-based Operational Modal Analysis (TOMA), Devriendt [16], uses transmissibilities to find modal parameters. The transmissibility is defined as the ratio of output i and a reference output j.

$$T_{ij}(\omega) = \frac{Y_i(\omega)}{Y_j(\omega)}$$

The transmissibility function converge to a constant value when $\omega \to \omega_n$

$$\lim_{\omega \to \omega_n} T_{ij}(\omega) = \frac{\phi_{i,n}}{\phi_{j,n}} = (K)$$

Thus under different loading conditions '*a*' and '*b*', the transmissibilities exactly cross at the structure resonance. Hence, the difference is zero when $\omega \to \omega_n$

$$\lim_{\omega \to \omega_n} T_{ij}^a(\omega) - T_{ij}^b(\omega) = \Delta T_{ij}^{ab} = 0$$

Thus the system poles correspond to the poles of inverse of the difference function.

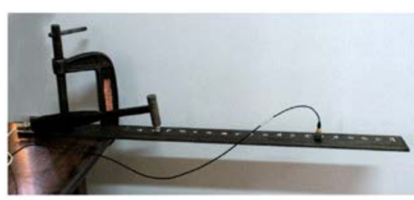
$$\frac{1}{\Delta T_{ij}^{ab}} = \frac{1}{T_{ij}^a(\omega) - T_{ij}^b(\omega)}$$

3 Experimental Set-up

The experimental setup used for time-data measurement at the response points is given in this section. The beam specifications are given Table 1.

Table 2: Beam specifications

Length(mm)	525
Width(mm)	50.2
Depth(mm)	6
Mass(kg)	1.566
Density(kg/m³)	9559.1



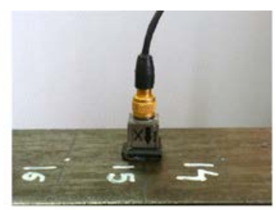


Figure 1: Experimental Setup and Accelerometer

The cantilever beam is divided in 20 equal parts. Numbering starts from the fixed end, which is fixed to the table using C-clamp, as shown in Figure 1. The accelerometer is attached at point number 15 by using beeswax. The accelerometer used is 3 axes type. It is attached by keeping its X-axis oriented in global Z-direction. B&K four channel FFT analyzer is used for data measurement. The sampling frequency used is 4096Hz.

Readings are taken using roving hammer technique, i.e. by hitting the beam with impact hammer at all the points. Four sets of readings are taken by changing the hammer tips. Four different tips are used - black, red, plastic and steel tip. The tip hardness increase in the following order - black, red, plastic and steel tips. The time data obtained is processed using MatLab code to find the mode parameters.

4 Results & Analysis

4.1 Analytical Method

The analytical values of the natural frequencies and the mode shapes for cantilever beam are given in this section. These values are used as benchmark results and are calculated using formulae discussed in section 2.1 Analytical Method. The obtained values for the natural frequencies are given in Table 3 and the analytical mode shapes are shown in Figure 2.

Table 3: Natural Frequencies of a Cantilever Beam

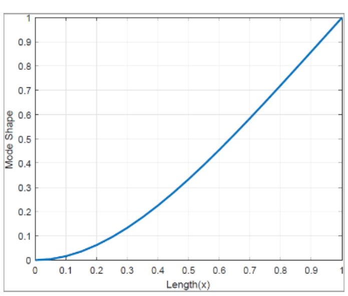
Mode (n)	ω_n (Hz)
1	16.482
2	103.29
3	289.22
4	566.76
5	936.89

4.2 Comparative Analysis - Analytical & EMA

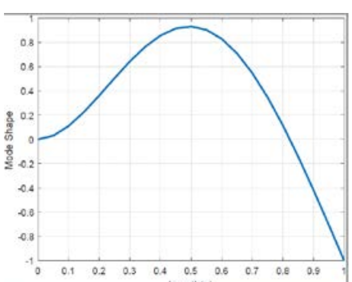
In this section the analytical natural frequencies and mode shapes are compared with the corresponding experimental values obtained using MatLab code for classical EMA. All the four datasets obtained using black, red, plastic and steel hammer tips are processed using classical EMA. For each mode the results from each dataset are compared with each other. This comparative study is used to finalize the dataset selection for further processing using OMA for studying a particular mode. The selected datasets are highlighted in Table 4. The comparative results are shown in Table 4. The frequency values are compared using the absolute error and the mode shapes are compared using Modal Assurance Criteria (MAC). The MAC value for two vectors is calculated as follows,

$$MAC(a,b) = \frac{\left|a^H b\right|^2}{\left(\left|a^H a\right|\right)\left(\left|b^H b\right|\right)}$$

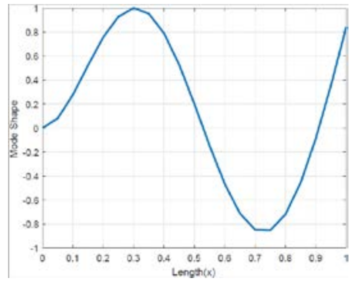
Where a^H is Hermitian transpose of vector a.



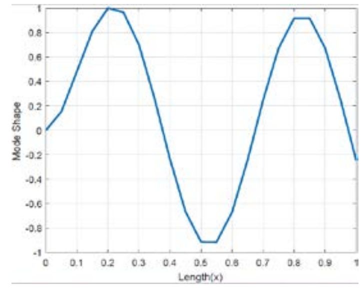
a) Mode 1



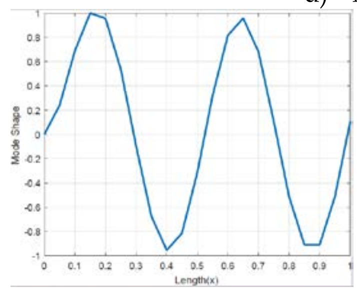
b) Mode 2



c) Mode 3



d) Mode 4



e) Mode 5

Figure 2: Analytical Mode Shapes

The blacktip database is selected for the first mode. Though the absolute error in the frequency value is less for the steel tip dataset, the MAC values are very low indicating bad prediction of the mode shapes. Hence the black tip dataset is selected for OMA study discussed in the 4.3 Comparative Analysis - EMA & OMA. The plastic tip dataset shows better prediction of the frequency values, but blacktip dataset is selected. The higher error (27.19%) in frequency value calculated for the black tip dataset, is due to the fact that the sampling frequency used for data collection is 4096Hz, thus the frequency step was 4Hz. If a smaller sampling frequency is used this error can be improved. However, the higher modes may not be obtained hence 4096Hz is used.

Table 4: Comparative Analysis - Analytical & EMA

Method	Mode 1		
	Freq(Hz)	% error	MAC
Analytical	16.48	NA	NA
Black Tip - EMA	12.00	-27.19%	95.48%
Red Tip - EMA	12.00	-27.19%	95.51%
Plastic Tip - EMA	13.00	-21.13%	96.39%
Steel Tip - EMA	13.25	-19.61%	27.03%
Method	**Mode 2**		
	Freq(Hz)	% error	MAC
Analytical	103.29	NA	NA
Black Tip - EMA	98.60	-4.54%	82.38%
Red Tip - EMA	96.00	-7.06%	96.65%
Plastic Tip - EMA	93.65	-9.33%	92.22%
Steel Tip - EMA	101.30	-1.93%	87.51%
Method	**Mode 3**		

	Freq(Hz)	% error	MAC
Analytical	289.22	NA	NA
Black Tip - EMA	297.40	2.83%	91.89%
Red Tip - EMA	248.00	-14.25%	79.68%
Plastic Tip - EMA	296.65	2.57%	96.77%
Steel Tip - EMA	294.25	1.74%	98.28%
Method	**Mode 4**		
	Freq(Hz)	% error	MAC
Analytical	566.76	NA	NA
Black Tip - EMA	548.40	-3.24%	14.39%
Red Tip - EMA	504.80	-10.93%	53.16%
Plastic Tip - EMA	542.85	-4.22%	87.84%
Steel Tip - EMA	525.55	-7.27%	85.54%
Method	**Mode 5**		
	Freq(Hz)	% error	MAC
Analytical	936.89	NA	NA
Black Tip - EMA	852.40	-9.02%	20.40%
Red Tip - EMA	802.40	-14.35%	24.03%
Plastic Tip - EMA	856.00	-8.63%	66.48%
Steel Tip - EMA	860.10	-8.20%	77.01%

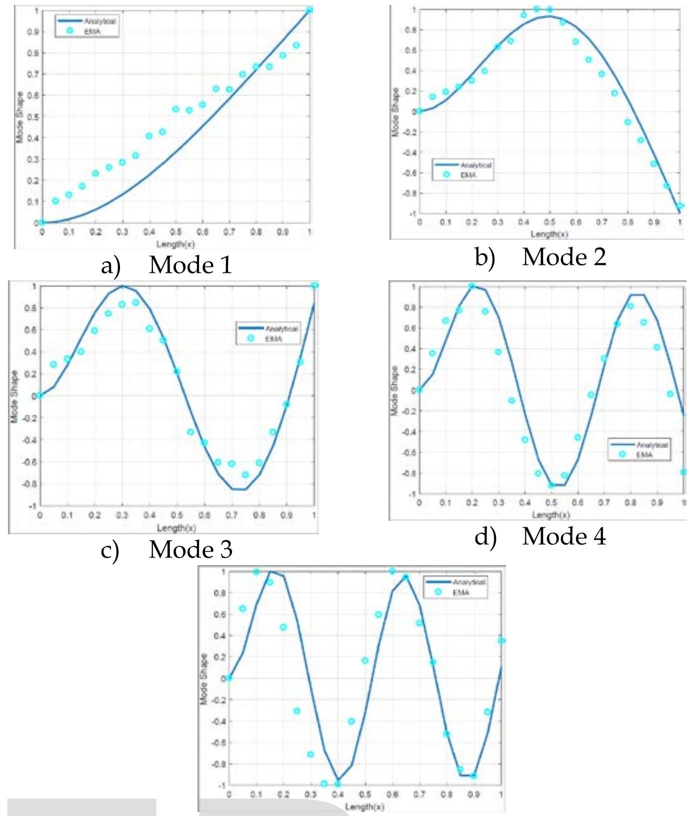


Figure 3:Mode Shapes - Analytical & EMA

In case of second mode, the best results are obtained for red tip dataset. The absolute error in frequency prediction is little higher but the MAC value (96.65%) is in best agreement. Similarly, for the third mode the plastic tip dataset MAC value (96.77%) is in best agreement, though steel tip dataset also give good results. The plastic tip dataset gives best results for the fourth mode (MAC value 87.84%) and the fifth mode is prediction is best using the steel tip dataset, (MAC value 77.01%).

It is clear that the softer tips identify the lower modes and as the hardness increases, the higher modes are identified. The dataset selection for each mode is given in Table 5.

Table 5: Database Selection - OMA

Mode	**Database**
Mode 1	Black Tip Database
Mode 2	Red Tip Database
Mode 3	Plastic Tip Database
Mode 4	Plastic Tip Database
Mode 5	Steel Tip Database

The analytical mode shapes are compared with the mode shapes obtained using EMA. The mode shapes only for the selected dataset for each mode are shown in Figure 3.

4.3 Comparative Analysis - EMA & OMA

This section gives comparative analysis of the frequency and mode shape results obtained using EMA and OMA. Using the comparative study from 4.2 Comparative Analysis - Analytical & EMA, Table 4, the highlighted datasets are selected for OMA. The list of selected dataset for each mode is given in Table 5.

The selected database for each mode is processed using all the five OMA techniques discussed in 2.3 Operational Modal Analysis (EMA). A comparative analysis for each mode using all the five techniques for OMA is given in Table 6. MAC is used for comparison of mode shapes obtained using EMA and OMA.

Table 6: Comparative Analysis - EMA & OMA

Method	**Mode 1 - (Black Tip)**		
	Freq(Hz)	abs error	MAC
EMA	12.00	NA	NA
OMA-BFD(PP)	16.00	33.33%	96.94%
OMA-FDD	16.00	33.33%	95.97%
OMA-ERA	12.98	8.20%	96.41%
OMA-LSCE	14.51	20.89%	96.94%
OMA-TOMA	16.00	33.33%	96.94%
Method	**Mode 2 - (Red Tip)**		
	Freq(Hz)	abs error	MAC
EMA	96.00	NA	NA
OMA-BFD(PP)	92.00	4.17%	85.15%

OMA-FDD	92.00	4.17%	75.68%
OMA-ERA	109.00	13.54%	68.39%
OMA-LSCE	109.98	14.56%	68.39%
OMA-TOMA	116.00	20.83%	38.71%
Method	**Mode 3 - (Plastic Tip)**		
	Freq(Hz)	abs error	MAC
EMA	296.65	NA	NA
OMA-BFD(PP)	248.00	16.40%	79.52%
OMA-FDD	248.00	16.40%	79.27%
OMA-ERA	240.58	18.90%	80.07%
OMA-LSCE	299.51	0.97%	82.06%
OMA-TOMA	336.00	13.26%	36.36%
Method	**Mode 4 - (Plastic Tip)**		
	Freq(Hz)	abs error	MAC
EMA	542.85	NA	NA
OMA-BFD(PP)	543.00	0.03%	92.17%
OMA-FDD	543.00	0.03%	90.80%
OMA-ERA	644.04	18.64%	20.95%
OMA-LSCE	543.46	0.11%	91.95%
OMA-TOMA	646.00	19.00%	21.04%
Method	**Mode 5 - (Steel Tip)**		
	Freq(Hz)	abs error	MAC
EMA	860.10	NA	NA
OMA-BFD(PP)	773.00	10.13%	79.92%
OMA-FDD	773.00	10.13%	55.44%
OMA-ERA	974.62	13.31%	0.36%
OMA-LSCE	941.90	9.51%	0.25%
OMA-TOMA	1104.00	28.36%	1.80%

Using the comparative analysis given in Table 6, it is observed that the modal parameters for the first mode are predicted using all the five techniques with good accuracy. The higher error (33%) in the natural frequency prediction is due to the sampling frequency selection, which is already discussed in the 4.2 Comparative Analysis - Analytical & EMA.

The absolute error in the natural frequency for the second mode is reduced as compared to the first mode. The maximum error observed is (20.83%), but if the obtained natural frequency values are compared with the analytical frequency value (103.29Hz), the maximum absolute error reduces to (12.30%). Except for the TOMA technique, all other techniques predict the mode shapes with reasonable accuracy.

In case of third mode the maximum absolute error in the natural frequency is obtained using ERA technique (18.90%). When compared to the analytical frequency value the maximum absolute error reduces to 16.82%. Similar to the second mode, except for the TOMA technique, all other techniques predict the third mode shapes with reasonable accuracy.

The fourth modal parameters, obtained using the plastic tip dataset, are predicted with very good accuracy using the BFD, FDD and LSCE techniques. The ERA and TOMA techniques perform very poorly in case of the fourth mode.

The fifth natural frequency is predicted with reasonable accuracy except using TOMA technique (maximum absolute error 28.36%). However, the mode shape prediction is poor in most of the techniques except BFD technique. The mode shape obtained using FDD though having greater error, is considered as the higher error is contributed due to variation near the free end of the cantilever beam.

The mode shapes for the highlighted OMA techniques in Table 6, are shown in Figure 4. These mode shapes are compared with those obtained using EMA.

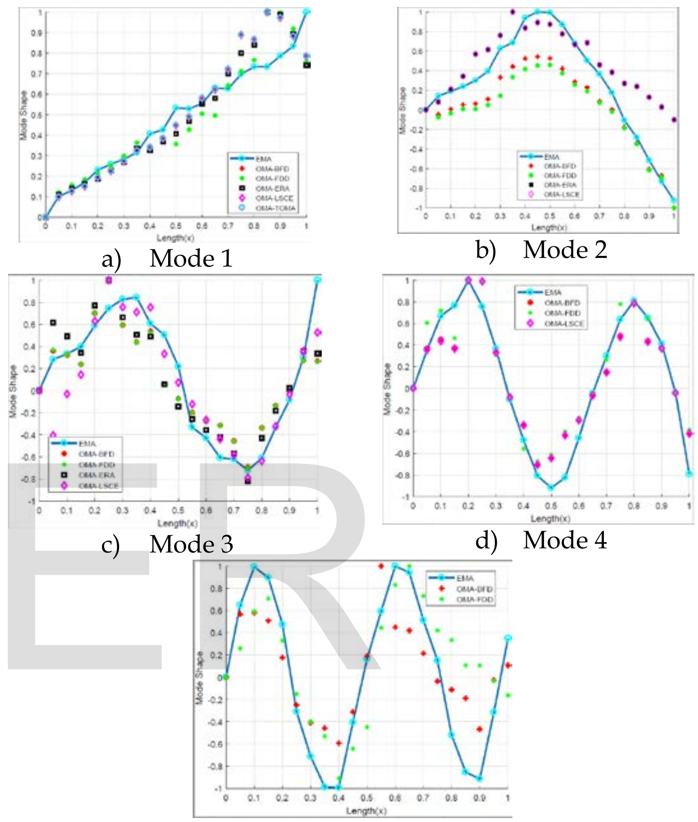


Figure 4: Mode Shapes - EMA & OMA

Though BFD and FDD techniques are simple techniques, both the techniques can be used for predicting all the five modes of the considered cantilever beam. These are numerically less intensive methods. The ERA technique works well in predicting lower modes. For higher modes though the natural frequencies are predicted with reasonable accuracy, the predicted mode shapes shown very high error. The time domain LSCE technique give very good accuracy in predicting the natural frequencies. Except for the fifth mode shape, all the other mode shapes are predicted with good accuracy using LSCE technique. TOMA technique works well only in case of the first mode. For all the other modes high error is obtained.

5 Conclusions

In the comparative study conducted for EMA and analytical results it is concluded that EMA technique identify all the modes. The accuracy of the results for lower modes is good for dataset obtained using soft tip hammer such as black tip, and that for higher modes is good for dataset obtained using hard tip hammer such as steel tip. Best results for EMA are taken for further comparison with OMA.

The results from five different OMA techniques are studied for each mode. The results obtained using OMA are compared with EMA using absolute error for the natural frequency and MAC values for the mode shapes. It is observed that all the five OMA techniques give accurate result for the first mode shape (least MAC value 95.97%). For the second and third modes, except for the TOMA technique, all the remaining four techniques (BFD, FDD, ERA and LSCE) give reasonably accurate results (least MAC value 68.39%). The ERA method fails to identify the fourth and fifth mode shape. The fifth mode shape is identified by the BFD and FDD methods.

All the five methods identify the natural frequencies with reasonable accuracy. The higher absolute error (33.33%) in the first natural frequency can be improved by changing the sampling frequency during data measurement. The OMA techniques can thus replace EMA, especially in situations where exciting the structure using a hammer or a shaker is not possible or in case where operating and laboratory boundary conditions are different. OMA can thus be used for the modal parameter extraction of the overhanging machine components where the data collected under exact operating conditions is important for better accuracy in the modal parameter extraction.

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